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Degeneracy in the particle-in-a-box problem

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Abstract. The notions of systematic and accidental degeneracy are discussed for a quantum-mechanical particle in a two-dimensional box. Separability of coordinates is shown to admit extra systematic degeneracy over that expected from the point-group symmetry. The 'accidental' degeneracy for square and triangular boxes is discussed in detail and related to geometric symmetries—described by rings of operators, however, and not by a symmetry group.

1. Introduction

Particle-in-a-box problems are the ABC's of quantum mechanics. They receive (brief) mention in almost every elementary textbook. Nonetheless, they can exhibit a startling degree of 'accidental' degeneracy, which is normally even when recognized, dismissed as either intractable or uninteresting (but see McIntosh 1968). The present paper demonstrates that, at least for some simple kind of 'boxes', these degeneracies are neither. Further, they provide examples in which a ring, rather than a group, must be introduced to deal with questions of degeneracy in a quantum-mechanical problem.

In two dimensions, the problem of a particle confined in a square box is highly degenerate. From the conventional viewpoint, the energy levels exhibit both 'systematic' degeneracy, apparently due to the symmetry of the box, and 'accidental' degeneracy which is not due to any obvious symmetry. Sections 2 and 3 discuss this problem in detail—§ 2 dealing with the 'systematic' degeneracies, while § 3 treats the 'accidental' ones. The method of § 3 is not restricted to separable problems, and § 4 discusses the case of a particle in an equilateral triangular box. Finally, § 5 summarizes the main conclusions and discusses the range of applicability both of the method and of the general approach.

2. The square box—systematic degeneracy

The particle in a square box is no stranger to anyone with even the briefest exposure to quantum mechanics. Two different problems are often considered with this name, and for both the solutions are easily obtained. For either, the differential equation to be satisfied is the free-particle equation

$$\Delta\Psi = E\Psi, \tag{1}$$

but two choices of boundary conditions are common. For convenience, we will consider a unit square bounded by the lines $x = 0, x = \pi, y = 0, y = \pi$. Then the conditions are:

BC1 $\quad \Psi(x, y) = 0$ on the boundary;

BC2 $\quad \Psi(x, y)$ is periodic in x, y with period 2π.

Of these two boundary conditions, BC2 are more general, in the sense that one can obtain any solution satisfying BC1 as a linear combination of degenerate solutions satisfying BC2. The converse, of course, is not true. This section will nonetheless discuss the problem under BC1, since the analysis is identical for the two problems but the greater number of solutions under BC2 would make the notation cumbersome.

Under BC1, then, equation (1) separates; the general solution and corresponding eigenvalue being,

$$\psi_{m,n} = \sin mx \sin ny \qquad E_{m,n} = m^2 + n^2.$$

Examining the eigenvalues, we see that $E_{m,n} = E_{n,m}$: the solution obtained by interchanging the x and y axes is degenerate with the original solution. Since this interchange is one of the geometric symmetries of the square, we will begin discussing the systematic degeneracy by examining the behaviour of the eigenfunctions under this group of operators.

The geometric symmetries for a square include rotations through multiples of 90°, reflection through the diagonals, and reflection through lines parallel to two sides and midway between them. The character table for this group, C_{4v}, is given as table 1.

Table 1. Character table for the group C_{4v}.

(C_{4v})	I	R^2	$2R$	$2\sigma_h$	$2\sigma_d$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
B_1	1	1	−1	1	−1
B_2	1	1	−1	−1	1
E	2	−2	0	0	0

Four non-degenerate representations and a single doubly-degenerate representation exist—presumably corresponding to wavefunctions of the types $\psi_{n,n}$ and $\psi_{m,n}$ respectively. Figure 1, however, shows the first few eigenfunctions and their classification under C_{4v}.

Lines indicate nodal lines for the functions. (The corresponding solutions to the particle in a circular box are also included, for comparison.) The first three levels are as expected, being A_1, E and B_2 symmetries respectively. The fourth, however, is of the reducible representation $A_1 + B_1$, a fact masked by the functions' usual separated form. The equivalent functions, $\sin 3x \sin y \pm \sin x \sin 3y$ (also shown in figure 1) which are of A_1 and B_1 symmetry respectively, show clearly that this degeneracy is not 'due to' the C_{4v} symmetry. This is again brought home when we note that the corresponding circular box functions belong to quite different levels. Thus in going from the $(C_{\infty h})$ circular box to the 'broken' symmetry square (C_{4v}), we see some levels split (eg $m = 2 \rightarrow B_1, B_2$) yet others coalesce ($m = 2$, B_1; $m = 0, A_1 \rightarrow (3, 1)$). This is not an

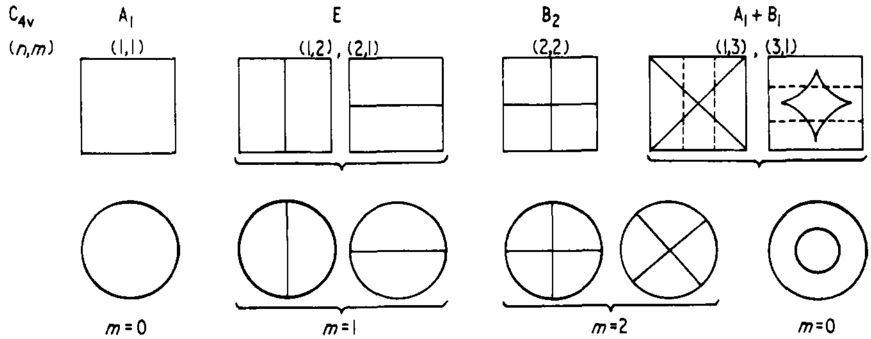

Figure 1. Lowest wavefunctions for particle in square and circular boxes. Interior lines indicate nodes, brackets set off degenerate pairs. Broken interior lines indicate an alternative choice of the degenerate pair.

isolated instance, as examination of the rest of the spectrum yields the following classification of states:

$\Psi_{n,n}$ n odd $= \mathrm{A}_1$

$\Psi_{n,n}$ n even $= \mathrm{B}_2$

$\Psi_{m,n}$ odd, even $= \mathrm{E}$

$\Psi_{m,n}$ both odd $= \mathrm{A}_1 + \mathrm{B}_1$

$\Psi_{m,n}$ both even $= \mathrm{A}_2 + \mathrm{B}_2$.

The $\mathrm{C}_{4\mathrm{v}}$ symmetries do not alone account for the degeneracy of $\Psi_{n,m}$ and $\Psi_{m,n}$ since this would require that any linear combinations of the functions would be mixed by operations in the group. As this is not the case, we must now look more closely at the problem.

If the 'systematic' degeneracy is to be related to the symmetry of the problem but the simple point-group symmetries of the potential are not adequate, what other symmetry have we so far ignored? We observe that the problem, as posed, is separable into two identical one-dimensional problems. This is indirectly reflected in the symmetries already noted, but the consequences should be examined directly.

The one-dimensional component of the problem consists of the differential equation

$$H_x\Psi = -\frac{1}{2}\frac{\partial^2}{\partial x^2}\Psi = -\tfrac{1}{2}\epsilon\Psi$$

with $\Psi(0) = \Psi(\pi) = 0$. The geometrical symmetry group for the problem consists of the identity I, and the reflection $m: x \to \pi - x$. Introducing two such systems, one in x and one in y, and a coordinate interchange operator a, we can find all the symmetries which arise from applying the basic symmetry operators successively:

$$I = I, \qquad \sigma_{\mathrm{d}} = a, \qquad \sigma_{\mathrm{d}'} = mam, \qquad \sigma_{\mathrm{h}} = m, \qquad \sigma_{\mathrm{h}'} = ama$$

$$m^2 = \sigma^2 = I, \qquad R_4 = am, \qquad R_4^3 = ma, \qquad R_4^2 = (am)^2 = (ma)^2.$$

Every product of the a, m and I operators can be shown to be equal to one of the above, which can be identified with the operators of $C_{4\mathrm{v}}$. Thus, the symmetry already determined, and no more, arises from the geometric symmetry of the one-dimensional problem together with the fact that the problem of interest is the direct product of two such identical systems.

There is however another label that can be associated with the one-dimensional problem—the eigenvalue ϵ of H. While this provides no information about degeneracies in one dimension, when the product of two such systems is considered these labels are no longer trivial. In particular, such states as $\sin x \sin 3y \pm \sin y \sin 3x$, which we have shown to be unmixed by any geometric symmetry, satisfy

$$(H_x - H_y)(\sin x \sin 3y + \sin y \sin 3x) = -2(\sin x \sin 3y - \sin y \sin 3x),$$

the A_1 state going over to the B_1 state (and conversely). Thus, the existence of H_x and H_y as 'symmetry' operators commuting with the full hamiltonian does complete the description of the degeneracy scheme we have observed. However, for each of H_x, H_y and $(H_x - H_y)$ all the integral powers are linearly independent, so any group with any of these operators as elements must be infinite dimensional. While such groups have been studied to some extent, their representations are not nearly so accessible as those of finite groups. There exists a finite group sufficient for our purposes, however. Define an operator Λ by expanding Ψ in eigenfunctions of H_x and H_y (H, H_x and H_y commute) as $\Psi = \Sigma a_{jk}\Psi_{jk}$ and writing

$$\Lambda\Psi = \sum a_{jk} u(\epsilon_j, \epsilon_k)\Psi_{jk}$$

where

$$u(x, y) = \begin{cases} 1 & x \geqslant y \\ -1 & x < y. \end{cases}$$

For states Ψ of finite multiplicity, $\Lambda\Psi$ remains differentiable and continuous and still satisfies the boundary conditions, so Λ is an operator on the H eigenfunctions of any given energy. In particular

$$\Lambda(\sin mx \sin ny \pm \sin nx \sin my) = u(m, n)(\sin mx \sin ny \mp \sin nx \sin my)$$

$$\Lambda(\sin mx \sin my) = \sin mx \sin my.$$

Thus Λ provides the mixing we are looking for. Since $\Lambda^2 = (\Lambda m)^2 = (\Lambda a)^2 = I$, the group generated by the symmetries (I, m, a, Λ) is finite. The group has 32 elements in 14 classes, and has the character table given in table 2. Of the fourteen possible symmetry types, only A_1^+, B_2^+, E_3^+, E_1 and E_2 actually occur with BC1. With BC2, representations of types E_4^+ and E_3^- also occur, though a larger group is necessary for

Table 2. Contracted character table for the group generated by the symmetries (I, m, a, Λ). Each entry represents one line from table 1.

	C_{4v}	$2C_{4v'\epsilon}$		$C_{4v\epsilon\epsilon'}$
$A_1^{\pm}$	A_1	$\pm A_1$		A_1
$A_2^{\pm}$	A_2	$\pm A_2$		A_2
$B_1^{\pm}$	B_1	$\pm B_1$		B_1
$B_2^{\pm}$	B_2	$\pm B_2$		B_2
E_1	$A_1 + B_1$	0		$-A_1 - B_1$
E_2	$A_2 + B_2$	0		$-A_2 - B_2$
$E_3^{\pm}$	E	$\pm E$		E
$E_4^{\pm}$	E	σ_h	$\sigma_{h'}$	E
		± 2	∓ 2	

proper classification (as below). The remaining symmetries do not occur within the space of eigenfunctions of H.

The above description applies as well for the problem under the periodicity conditions we called BC2. The momentum, or 'infinitesimal displacement' operator is an additional symmetry for the latter case. Further, for $p_x = \mathrm{i}\partial/\partial x$, $p_x^2 = 2H_x$, so the energy label is redundant for momentum eigenstates. Again, p is unbounded, so as before we define a new operator which allows us to form a finite group under which every systematic degeneracy corresponds to an irreducible group representation. If $\Phi_{k,l}$ is an eigenfunction of p_x, p_y then as before, we define the operator

$$\Lambda'\Psi = \Lambda' \sum a_{k,l}\Phi_{k,l} = \sum a_{k,l}u(\kappa_k, \kappa_l)\Phi_{k,l}.$$

The group generated by (I, m, a, Λ') then, is finite and properly classifies the momentum wavefunctions. It is too large for convenient representation, however.

We can at last say that we have adequately described the 'systematic' degeneracy of the particle in a square box. It is seen to arise from the problem's being the product of two identical one-dimensional problems. This, together with the existence of a mirror plane in the one-dimensional system, suffices to classify states according to table 2. If in addition the momentum commutes with the one-dimensional hamiltonian, the further classification derived from Λ' applies.

3. The square box—accidental degeneracy

We began discussing degeneracy for the square box by classifying degeneracies as 'systematic' or 'accidental'. Having disposed of the systematic structure characteristic of any such direct product problem, we will examine the remainder, which appears to arise specifically from the constancy of $V(x, y)$ throughout the box. That there is something significant to explain may be seen by the following degeneracies $D(E)$:

$$D(50) = 3, \qquad \text{with } E_{7,1} = E_{1,7} = E_{5,5}.$$

$$D(1105) = 8, \qquad \text{with } E_{4,33} = E_{9,32} = E_{12,31} = E_{23,24} = E_{24,23} = \dots$$

$$D(5\,928\,325) = 48.$$

All these degeneracies should be compared with those discussed in the last section, where $D(E_{m,m}) = 1$ and $D(E_{n,m}) = 2$ were the only degeneracies found to arise from the obvious symmetries.

In attacking this new problem, a fresh start seems necessary. In fact, to clarify the points to be made, we reformulate the problem slightly. With $z = x + \mathrm{i}y$,

$$\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 = \partial^2/\partial z\partial z^*$$

so we can write

$$h\Psi(z) = \partial^2/\partial z\partial z^* \,\Psi(z) = \lambda\Psi(z), \tag{1'}$$

subject to

$$\Psi(z+z_i) = \Psi(z) \qquad z_1 = 2\pi, \qquad z_2 = 2\pi i. \tag{2'}$$

Suppose that, in fact, a solution $\Psi_s(z)$ with eigenvalue λ_s has been found satisfying (1′) and (2′). We then see that for any complex constant α, $\Psi_s(\alpha z)$ is also a solution of (1′) with eigenvalue $\alpha^*\alpha\lambda_s$. However, this will not in general satisfy (2′). The conditions under which $\Psi_s(\alpha z)$ is a proper solution are easy to obtain. The requirement is:

$$\Psi_s(\alpha(z+z_i)) = \Psi_s(\alpha z+\alpha z_i) = \Psi_s(\alpha z) \qquad i = 1, 2.$$

This will be automatically the case if $\alpha z_i = \Sigma\, n_j z_j$ for integers n_j.

We can write

$$\alpha = a+b\mathrm{i} \qquad (a, b \text{ real}),$$

then

$$a + b\mathrm{i} = n_1+n_2\mathrm{i}$$

$$(a + b\mathrm{i})\mathrm{i} = -b+a\mathrm{i} = n_3+n_4\mathrm{i} \qquad n_i \text{ integers.}$$

We thus have the result:

For any solution $\Psi_s(z)$ to (1′) satisfying (2′), $\Psi_s^{a,b}(z) \equiv \Psi_s((a+b\mathrm{i})z)$ is also a proper solution for any integers a and b. If the eigenvalue of Ψ_s is λ_s, the eigenvalue of $\Psi_s^{a,b}$ is $(a^2+b^2)\lambda_s$.

In fact, if we can choose Ψ_s to be the state with smallest eigenvalue, then every such proper solution may be a $\Psi_s^{a,b}$.

Clearly, while this analysis has by no means solved the problem, it has identified a powerful means for discussing the spectrum, etc. Our interest, however, is specifically in degeneracies. Consider a state $\Psi_s^{a,b}$, where $a+b\mathrm{i} = (p_1+\mathrm{i}q_1)(p_2+\mathrm{i}q_2)$; p_i, q_i integers. One could obviously form $\Psi_s^{c,d}$ with $c+d\mathrm{i} = (p_1-\mathrm{i}q_1)(p_2+\mathrm{i}q_2)$. Further,

$$\lambda_{c,d} = \lambda_{a,b} = (p_1^2+q_1^2)(p_2^2+q_2^2)\lambda_s.$$

Thus, degeneracy of levels may arise whenever the label for a state may be so factored. But the theory of factorization of numbers of the type $m+n\mathrm{i}$ ('gaussian', or complex integers) has long been known. Every such integer can be shown to be one of four mutually exclusive types (Le Vegue 1956); 'units' comprise ± 1, $\pm\mathrm{i}$, 'conjugate primes' comprise $1\pm\mathrm{i}$, and the real 'complex' primes $3, 7, 19, \ldots$, whose complex conjugates are their unit multiples, 'normal primes' which are not multiples of their complex conjugates and include $2\pm\mathrm{i}$, $3\pm 2\mathrm{i}$, $4\pm\mathrm{i}, \ldots$, 'composite numbers' such as

$$5 = (2+\mathrm{i})(2-\mathrm{i}), \qquad (3+\mathrm{i}) = (2-\mathrm{i})(1+\mathrm{i}), \ldots.$$

One can then state:

Factorization theorem (Gauss): Every gaussian integer k can be expressed as a product $k = \Pi_{i=1}^{n}\, p_i$ of prime gaussian integers. This representation is essentially unique; ie any other decomposition of k into primes has the same number of factors and can be so rearranged that corresponding factors are associates (unit multiples).

We can see, then, that if $a+b\mathrm{i}$ has the prime decomposition $a+b\mathrm{i} = \Pi_{i,j}\, q_i^{s_i} \cdot p_j^{t_j}$ with q_i conjugate primes, p_j normal primes, that $D(\lambda_{a,b}) = \Pi_j(t_j+1)$. It is further clear, that the states degenerate with $\Psi^{a,b}$ are labelled by a', b'; a'', b''; etc formed by taking conjugates of the terms in the product of primes in every possible combination. For instance, consider $\Psi_s^{7,1}$.

$$(1-\mathrm{i})(2+\mathrm{i})(2+\mathrm{i}) = 7+\mathrm{i}$$

and

$$(1-\mathrm{i})(2+\mathrm{i})(2-\mathrm{i}) = 5-5\mathrm{i}.$$

(The other conjugated terms simply yield the 'systematic' degenerate terms.) Thus, the degeneracy $\lambda_{7,1} = \lambda_{5,5}$ mentioned at the beginning of the section can be obtained. Similarly, the decomposition $(33+4\mathrm{i}) = (2+\mathrm{i})(3-2\mathrm{i})(4+\mathrm{i})$ gives rise to the identities, $\lambda_{33,4} = \lambda_{32,9} = \lambda_{31,12} = \lambda_{24,23}$ also mentioned.

Finally, we note that a proper solution to (1′), (2′) is $\Psi_0(z) = \exp[\frac{1}{2}(z-z^*)]$. Writing $\Psi_{a,b} = \Psi_0^{a,b} = J_{a,b}\Psi_0$, we find that *every* solution to the problem is a $\Psi_{a,b}$. Thus, $\lambda_{a,b} = (a^2+b^2)\lambda_0 = a^2+b^2$, and the degeneracy described as $D(\lambda_{a,b})$ is the full degeneracy of the problem.

We thus have arrived at a systematic labelling and classification of the energy levels (via the vectors $(t_1, \ldots, t_k)$) which determines the associated degeneracies simply. Still, the reduction to a form where a theorem from number theory could be applied may seem very artificial—far removed from the spatial and permutation symmetries to which we normally ascribe degeneracy, or the group structure that normally allows classification. It is therefore important to make clear just what symmetry is being used in this classification, and to what extent the kind of approach used in this problem could be expected to be of use in general.

Because we have chosen periodic boundary conditions for our problem, we could consider our square 'box' not to be isolated, but rather to be one unit of an infinite two-dimensional lattice. The wavefunction on the square could also be considered to be defined throughout space by means of the periodicity condition. Each multiplication by a constant $a+b\mathrm{i}$ then produces a contraction of space by a factor $(a^2+b^2)^{1/2}$, combined by a rotation through an angle $\tan^{-1}(b/a)$, thereby bringing another (larger) unit square into the 'standard' square's position. The importance of the potential energy being constant, then, is that this is the only potential which is invariant under all such transformations of the plane.

The mathematical structure introduced is of some interest, and certainly is not one commonly used in mathematical physics. Defining step-up operators $J_{a,b}$ by

$$J_{a,b}\Phi(z) = \Phi((a+b\mathrm{i})z),$$

the $J_{a,b}$ are all related to a spatial symmetry. In fact, they are the contraction–rotation operators just discussed. However, they are neither symmetries (they do not commute with the hamiltonian) nor do they form a group (their inverses are not operators). Both of these differences from the group operations usually employed to describe degeneracies deserve special mention. First, the absence of inverses. The reason for this is easy to see. Any function which is periodic with a period 2π is obviously also periodic with period $2n\pi$. It is not in general periodic with a period $2\pi/n$. Thus contractions can be found which maintain periodicity on the lattice, while their inverses, dilations, cannot. The J's then, do not form a group. Nonetheless, any product of two such operators is another such operator. Further, though this is less obvious, the sum of any two operators can be defined, and is such an operator. They can thus be shown to form a commuting ring; in fact, a special ring called an integral domain. The latter two mathematical objects are defined in the appendix. Their importance is that they are well studied, and in particular, factorization theorems exist for any such sets of operators (see for example, Birkhoff and MacLane 1953). Thus, for any problem in which solutions can be generated from 'elementary' solutions, as for the particle in a square box, the same kind of factorization and conjugation device will give a procedure for finding states degenerate with the initial one. While the result need not always be so complete as in the present case, it can still be useful.

4. Degeneracy in the triangular box

In the last section we completely analysed, in an unusual way, a problem which is so readily 'solved' that the result may seem to be trivial. To demonstrate that separability of the equations does not affect the ease of solution, we will briefly discuss the problem of a particle in an equilateral triangular box by the same method.

As before, we wish to solve equations (1′), (2′), but now with:

$$z_1 = 2\pi, \qquad z_2 = \pi(1+\sqrt{3}\mathrm{i}), \qquad z_3 = \pi(1-\sqrt{3}\mathrm{i}).$$

In this case $a+b\mathrm{i}$ are determined by

$$a+b\mathrm{i}\,.\,1 = a+b\mathrm{i} = n_1+n_2(1+\mathrm{i}\sqrt{3})/2$$

$$(a+b\mathrm{i})(1+\mathrm{i}\sqrt{3})/2 = (a-b\sqrt{3})/2+(a\sqrt{3}+b)\mathrm{i} = n_3+n_4(1+\mathrm{i}\sqrt{3})/2.$$

Together, these equations imply that $a+b\mathrm{i} = m+n(1+\mathrm{i}\sqrt{3})/2$ with m and n integers. (As z_3 is an integral combination of z_1 and z_2, its inclusion provides no additional restriction.) The $a+b\mathrm{i}$ (or their associated operators), form an integral domain as did the gaussian integers. Again, every such 'integer' can be uniquely (except for units) factored into primes. In this case, the units are ± 1, $\pm(1+\mathrm{i}\sqrt{3})/2$, $\pm(1-\mathrm{i}\sqrt{3})2$; and the self-conjugate primes are the real primes and the unit multiples of $(3+\mathrm{i}\sqrt{3})/2$.

One has then,

$$\lambda^s_{m,n} = (m^2+mn+n^2)\lambda^s$$

$$D(\lambda^s_{m,n}) = \prod (s_i+1)$$

where the s_i are the exponents of the (unique) prime factors, as before.

Since this example is somewhat less familiar than the first, we will look briefly at some 'accidentally' degenerate levels in this system. Just as for the square box, levels of arbitrarily high degeneracy do occur (as indicated by the degeneracy function above), and just as for the square box the only degeneracy 'expected' is twofold degeneracy due to the rotational symmetry of the problem (C_{3v}). In this case, this systematic structure is properly accounted for by the geometric symmetry since the problem is not separable.

Again like the square box, all solutions to the triangular problem are obtained from a single elementary solution. The solutions satisfying BC1 are a combination of two such simple solutions. Explicitly:

$$\Psi_\alpha = \exp[\tfrac{1}{2}(\alpha z-\alpha^* z^*)]+\exp[\tfrac{1}{2}(\omega\alpha z-[\omega\alpha z]^*)]+\exp[\tfrac{1}{2}(\omega^2\alpha z-[\omega^2\alpha z]^*)]$$
$$-\exp[\tfrac{1}{2}(\alpha z^*-\alpha^* z)]+\exp[\tfrac{1}{2}(\omega\alpha z^*-[\omega\alpha]^* z)]-\exp[\tfrac{1}{2}(\omega^2\alpha z^*-[\omega^2\alpha]^* z)].$$

Thus,

$$\Psi_1 = \exp(\mathrm{i}y)+\exp[\mathrm{i}(-\tfrac{1}{2}y+\tfrac{1}{2}\sqrt{3}x)]+\exp[\mathrm{i}(-\tfrac{1}{2}y-\tfrac{1}{2}\sqrt{3}x)]-\exp(-\mathrm{i}y)$$
$$-\exp[\mathrm{i}(\tfrac{1}{2}y+\tfrac{1}{2}\sqrt{3}x)]-\exp[\mathrm{i}(\tfrac{1}{2}y-\tfrac{1}{2}\sqrt{3}x)]$$
$$= 2(\sin y-2\sin\tfrac{1}{2}y\cos\tfrac{1}{2}\sqrt{3}x)$$

$$\Psi_3 = 0 \qquad \text{(inconsistent with BC1)}$$

$$\Psi_4 = \sin 2y - 2\sin y \cos \sqrt{3}x$$

$$\Psi_7 = \begin{cases} \cos(\tfrac{1}{2}\sqrt{3}x)\sin(\tfrac{1}{2}5y) - \cos(\sqrt{3}x)\sin 2y - \cos(\tfrac{1}{2}3\sqrt{3}x)\sin\tfrac{1}{2}y \\ \sin(\tfrac{1}{2}\sqrt{3}x)\sin(\tfrac{1}{2}5y) - \sin(\sqrt{3}x)\sin 2y + \sin(\tfrac{1}{2}3\sqrt{3}x)\sin\tfrac{1}{2}y \end{cases}$$

$$\vdots$$

$$\Psi_{49} = \sin 7y - 2\sin(\tfrac{1}{2}7y)\cos(\tfrac{1}{2}7\sqrt{3}x)$$

$$\Psi'_{49} = \begin{cases} \cos(4\sqrt{3}x)\sin y - \cos(\tfrac{1}{2}3\sqrt{3}x)\sin(\tfrac{1}{2}13y) + \cos(\tfrac{1}{2}5\sqrt{3}x)\sin(\tfrac{1}{2}11y) \\ \sin(4\sqrt{3}x)\sin y + \sin(\tfrac{1}{2}3\sqrt{3}x)\sin(\tfrac{1}{2}13y) - \sin(\tfrac{1}{2}5\sqrt{3}x)\sin(\tfrac{1}{2}11y) \end{cases}$$

(the first 'accidentally' degenerate level).

5. Discussion and summary

In considering the range of applicability of the above approach, one can consider either what systems the specific method described can deal with, or what other problems may lend themselves to a solution of this general type (ie construction of a generating ring). The first question can be answered fairly easily.

In two dimensions, the only potential which is both scale invariant and compatible with periodic boundary conditions is the constant potential. Thus, only 'free' particles can be treated. Further, the only boundaries for which periodicity conditions necessarily yield solutions vanishing on the boundary are rectangular, and right and isosceles triangular ones. In each case, only particular ratios of sides will yield an 'integer like' ring and systematic accidental degeneracies. Those which do can be analysed as above.

In three or more dimensions (again for 'free' particles) the trick of writing the equation in complex form is not available. The 'constants' by which one transforms the space are therefore the affine transformations, which in n dimensions can be represented $n \times n$ matrices. The consistency relations then restrict one to a particular set (ring) of such matrices which, when systematic degeneracies occur, is again integer-like. For these cases the energy spectrum may be obtained, as well as the degeneracy of any given state. The corresponding factorization theorems are not so strong, but should allow the determination of symmetry labels analogous to the $(t_1, \ldots, t_k)$ found for the square and triangle problems, along with expressions for the degeneracy associated with each label. Again, n-dimensional parallelopipeds and simplexes (n-triangles) could be so treated.

The more interesting problem is that of obtaining a generative ring to analyse the degeneracy in problems where no physically intelligible (eg space or permutation group) accounts for it. No general criterion for determining when such an approach can be used is yet available. The similarity between the generating ring and a 'spectrum generating algebra' (SGA) is striking. The latter, finding increasing use in recent years (see, for example, Cordero and Ghirardi 1972) suggest that generating rings may become of utility in a larger and more intrinsically interesting set of quantum-mechanical problems.

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Appendix

Definition. A *ring* is a system of elements with two binary operations: addition and multiplication. It is an abelian group under addition (ie addition is associative and commutative, and there exists an additive identity and an additive inverse for every element. Multiplication is associative, and distributes with respect to addition. Thus, for every a, b, c in the ring A,

$$a(bc) = (ab)c, \qquad a(b+c) = ab+bc, \qquad (a+b)c = ac+bc.$$

Examples of rings are $n \times n$ matrices with the usual addition and multiplication, the polynomials in m independent variables, and the operators on a vector space with multiplication as successive application.

Definition. An *integral domain* is a ring in which multiplication is commutative, a multiplicative identity (unit) exists, and, for any element $c \neq 0$,

$$ca = cb \rightarrow a = b \qquad \text{(cancellation law).}$$

Examples of integral domains are the real and complex integers (all components integers) with normal addition and multiplication.

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